On arithmetic and geometry around the adelic Eisenstein function

June 29 , 2021

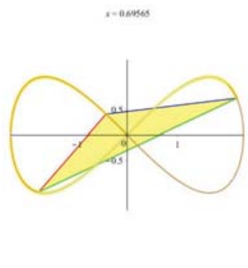
Hiroaki Nakamura (Osaka U)

[1] Introduction

[2] Focus on $M_{1,1}, M_{1,2}$

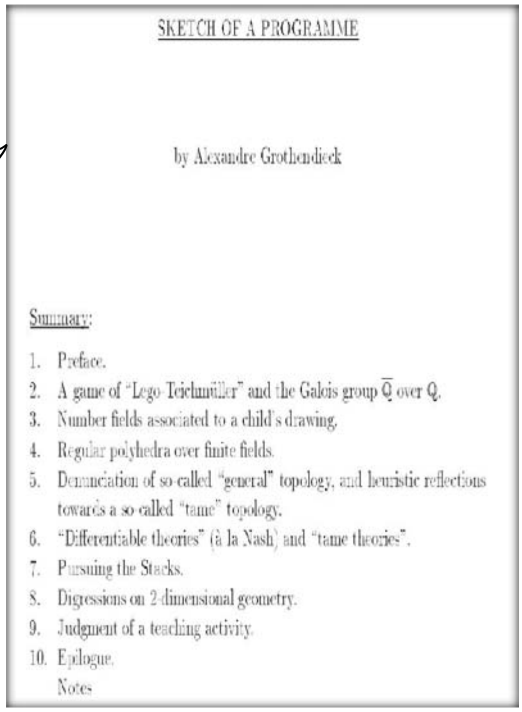
[3] Profinite Rademacher function

[4] Examples (Lissajous 3-braids)



1 Introduction

Grothendieck-Teichmüller theory



SKETCH OF A PROGRAMME

by Alexandre Grothendieck

Summary:

1. Preface.
2. A game of "Lego-Teichmüller" and the Galois group $\overline{\mathbb{Q}}$ over $\mathbb{Q}$.
3. Number fields associated to a child's drawing.
4. Regular polyhedra over finite fields.
5. Denunciation of so-called "general" topology, and heuristic reflections towards a so-called "tame" topology.
6. "Differentiable theories" (à la Nash) and "tame theories".
7. Pursuing the Stacks.
8. Digressions on 2-dimensional geometry.
9. Judgment of a teaching activity.
10. Epilogue.

Notes

Esquisse du Programme 1984

$M_{g,n}$: moduli space (stack) of the smooth proj. curves of genus g with n marked points.

$2-2g-n<0$ (hyperbolic condition)

fundamental object:

$$1 \longrightarrow \underset{\substack{\Uparrow \\ \text{profinite completion} \\ \text{of Teichmüller modular gp} \\ \text{(mapping class group)}}}{\hat{\Gamma}_{g,n}} \rightarrow \underset{\substack{\Uparrow \\ \text{profinite etale} \\ \text{fund. gp}}}{\pi_1(M_{g,n}/\mathbb{Q})} \longrightarrow G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1$$

There are many kinds of morphisms between $\left\{ \begin{array}{c} M_{g,n} \cup \partial M_{g,n} \\ \nwarrow \quad \nearrow \\ \text{smaller } M_{\gamma,\nu} \end{array} \right\}$

Deligne-Mumford compactification

Special loci, e.g. hyperelliptic locus (genus "zero"), other loci (curves with many automorph.)

Schneps, Tsunogai, Collas, ...

$\boxed{\text{Lego of Galois-Teichmüller tower}}$

Lego of Galois-Teichmüller tower

The collection

$$\left\{ \begin{array}{c} \pi_1(M_{g,n}) \\ \downarrow \\ G_{\mathbb{Q}} \end{array} \right\}_{2-2g-n<0}$$

forms a system of profinite groups

various $G_{\mathbb{Q}}$-compatible connections

fundamental blocks for the groupoid tower

"2-level principle"

$M_{0,3} = \mathrm{Spec}\,\mathbb{Q}$

$M_{0,4} = \mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}$

$M_{0,5} = \mathbb{P}^2_{\mathbb{Q}} - \quad = \mathrm{Conf}^2(M_{0,4})$

$M_{1,1} =$ "fine j-line"

$M_{1,2} =$ universal elliptic curves

+ suitable system of (tangential) base pts

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Geometric Galois Actions

Edited by Leila Schneps, Pierre Lochak

I considered some concrete cases (for coverings of low degree) by various methods, J. Malgoire considered some others – I doubt that there is a uni-
13 14 form method solving the problem by computer. My reflection quickly took a more conceptual path, attempting to apprehend the nature of this action of $\mathbb{\Gamma}$. One sees immediately that roughly speaking, this action is expressed by a certain "outer" action of $\mathbb{\Gamma}$ on the profinite compactification of the oriented cartographic group C_2^+, and this action in its turn is deduced by passage to the quotient of the canonical outer action of $\mathbb{\Gamma}$ on the profinite fundamental group $\hat{\pi}_{0,3}$ of $(U_{0,3})_{\overline{\mathbb{Q}}}$, where $U_{0,3}$ denotes the typical curve of genus 0 over the prime field $\mathbb{Q}$, with three points removed. This is how my attention was drawn to what I have since called "anabelian algebraic geometry", whose starting point was exactly a study (limited for the moment to characteristic zero) of the action of "absolute" Galois groups (particularly the groups $\mathrm{Gal}(\overline{K}/K)$, where K is an extension of finite type of the prime field) on (profinite) geometric fundamental groups of algebraic varieties (defined over K), and more particularly (breaking with a well-established tradition) fundamental groups which are very far from abelian groups (and which for this reason I call "anabelian"). Amond these groups, and very close to the group $\hat{\pi}_{0,3}$, there is the profinite compactification of the modular group $\mathrm{Sl}(2,\mathbb{Z})$, whose quotient by the centre ± 1 contains the former as congruence subgroup mod 2, and can also be interpreted as an oriented "cartographic" group, namely the one classifying triangulated oriented maps (i.e. those whose faces are all triangles or monogones).

It was only close to three years later, seeing that decidedly the vast horizons opening here caused nothing to tremble in any of my students, nor even in any of the three or four high-flying colleagues to whom I had occasion to talk about it in a detailed way, that I made a first reconnoitering voyage into this "new world", from January to June 1981. This first try materialized into a packet of some 1300 handwritten pages, baptized "The Long March through Galois theory". It is first and foremost an attempt at understanding the relations between "arithmetic" Galois groups and profinite "geometric" fundamental groups. Quite quickly it became oriented towards a work of computational formulation of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{\pi}_{0,3}$, and in a later stage, on the somewhat larger group $\widehat{\mathrm{Sl}(2,\mathbb{Z})}$, which gives rise to a more elegant and efficient formalism. Also during the course of this work
15/16 (but developed in a different set of notes) appeared the central theme of anabelian algebraic geometry, which is to reconstitute certain so-called "anabelian" varieties X over an absolute field K from their mixed fundamental group, the extension of $\mathrm{Gal}(\overline{K}/K)$ by $\pi_1(X_{\overline{K}})$; this is when I discovered the "fundamental conjecture of anabelian algebraic geometry", close to the conjectures of Mordell and Tate recently proved by Faltings ([3]). This is also where I began a first reflection on the Teichmüller groups, and the first intuitions on the multiple structure of the "Teichmüller tower" – the open modular multiplicities $M_{g,\nu}$ also appearing as the first important examples in dimension > 1, of varieties (or rather, multiplicities) seeming to deserve the appellation of "anabelian". Towards the end of this period of reflection, it appeared to me to be a fundamental reflection on a theory still completely up in the air, for which the name "Galois-Teichmüller theory" seems to me more appropriate than the name "Galois Theory" which I had at first given to my notes.

Ex. Forgetting one marked point induces

$$1 \longrightarrow \hat{\Pi}_{g,n} \longrightarrow \pi_1(M_{g,n+1}) \longrightarrow \pi_1(M_{g,n}) \longrightarrow 1$$

$\parallel$

profinite completion of $\Pi_{g,n} = \pi_1($ genus g surface with n punctures $)$

$2-2g-n<0 \Leftrightarrow$ far from abelian group (anabelian)

& induces the universal monodromy representations

$$\varphi_{g,n} : \pi_1(M_{g,n}) \longrightarrow \mathrm{Out}(\hat{\Pi}_{g,n})$$

- profinite analog of Dehn-Nielsen mapping
- injectivity (Belyi, Asada, Matsumoto-Tamagawa, Boggi ...)
- Combin. description of $\mathrm{Im}(\varphi_{g,n}) \approx$ "$\widehat{GT}_{g,n}$" $\subset \mathrm{Out}(\hat{\Pi}_{g,n})$ (Drinfeld, Ihara, Enriquez, ..)

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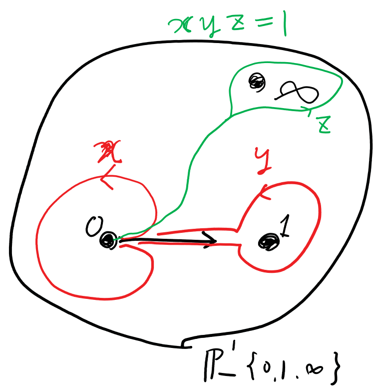
gles or monogones).

Ihara discovered much deeper arithmetic phenomenon in

$$1 \longrightarrow \hat{F}_2 \longrightarrow \pi_1\left(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \vec{01}\right) \longrightarrow G_{\mathbb{Q}} \longrightarrow 1$$

$\vec{01}$

$\hat{F}_2$ = free profinite group gen. by x, y



$$\varphi_{0,3} : G_{\mathbb{Q}} \hookrightarrow \widehat{GT} = \left\{ \alpha \in \mathrm{Aut}\hat{F}_2 \;\middle|\; \begin{array}{l} \exists!\,(\chi, f) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2' \text{ s.t.} \\ \alpha(x) = x^\chi \\ \alpha(y) = f^{-1} y^\chi f \\ \alpha(z) \underset{\mathrm{conj}}{\sim} z^\chi \end{array} \right.$$

$\widehat{GT} \subset \mathrm{Aut}\,\hat{\Pi}_{0,3}$

satisfying three combinatorial relations

(I) $f(x,y) = f(y,x)^{-1}$

(II) $f(x,z)\, x^{\frac{\chi-1}{2}} f(z,x)\, z^{\frac{\chi-1}{2}} f(y,z)\, y^{\frac{\chi-1}{2}} = 1$

(III) "Pentagon relation"

$\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times$: $\boxed{\text{cyclotomic character}}$ def. by $G_{\mathbb{Q}} \circlearrowright \{\zeta_n = \exp(\frac{2\pi i}{n})\} \subset \bar{\mathbb{Q}} \underset{\text{fix}}{\subset} \mathbb{C}$

$f : G_{\mathbb{Q}} \to \hat{F}_2' = [\hat{F}_2, \hat{F}_2] \subset \hat{F}_2$: $\boxed{\text{main parameter}}$ (big non-abelian cocycle

$G_{\mathbb{Q}} \circlearrowright_{\varphi_{0,3}}$

Meta-abelian reduction

$$(G_{\mathbb{Q}} \subset)\ \widehat{GT} \xrightarrow{\varphi_{0,3}} \mathrm{Aut}(\hat{F}_2) \longrightarrow \mathrm{Aut}(\hat{F}_2/\hat{F}_2'')$$

is encoded ~~by~~ in the adelic beta function of Anderson-Ihara theory

$$\widehat{GT} \ni \sigma \longmapsto B_\sigma(\bar{x}, \bar{y}) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]^\times = \hat{\mathbb{Z}}[[\hat{F}_2^{ab}]]^\times$$

- Combinatorial description using profinite free diff. calculus
- $B_\sigma(\zeta_n^a, \zeta_n^b) \in \hat{\mathbb{Z}}^\times$ $(\sigma \in G_{\mathbb{Q}})$ interpolates Jacobi sum Hecke characters
- Coefficients of B_σ in finite group rings $\hat{\mathbb{Z}}[(\mathbb{Z}/m\mathbb{Z})^2]$ give

Kummer-Soulé characters

$$\widehat{GT} \underset{K_m^{(a)}}{\overset{\rho_m}{\rightrightarrows}} \hat{\mathbb{Z}} \quad \text{extending actions of } G_{\mathbb{Q}} \circlearrowright \left\{ \sqrt[N]{m},\ \sqrt[N]{(1-\zeta_m^a)} \right\}$$

$(1 \leq a \leq m-1)$

Summary

$$\begin{tikzcd}
 & & \hat{\mathbb{Z}}^{\times} \\
G_{\mathbb{Q}} \arrow[r, hook] & \widehat{GT} \arrow[ur, "\chi"] \arrow[r, "f"'] \arrow[dr, "B"'] \arrow[ddr, "\rho_m,\ \kappa_m^{(a)}"'] & \hat{F}_2' \subset \hat{F}_2 \\
 & & \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]^{\times} \\
 & & \hat{\mathbb{Z}}
\end{tikzcd}$$

cyclotomic character

main parameter (Drinfeld-Ihara associator)

adelic beta function

arithmetic Kummer characters

$\boxed{2}$ Focus on $M_{1,1}$, $M_{1,2}$

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Geometric Galois Actions

Edited by Leila Schneps, Pierre Lochak

I would like to conclude this rapid outline with a few words of commentary on the truly unimaginable richness of a typical anabelian group such as $\mathrm{Sl}(2,\mathbb{Z})$ – doubtless the most remarkable discrete infinite group
$\frac{16}{17}$ ever encountered, which appears in a multiplicity of avatars (of which certain have been briefly touched on in the present report), and which from the point of view of Galois-Teichmüller theory can be considered as the fundamental "building block" of the "Teichmüller tower". The element of the structure of $\mathrm{Sl}(2,\mathbb{Z})$ which fascinates me above all is of course the outer action of Γ on it profinite compactification. By Bielyi's theorem, taking the profinite compactifications of subgroups of finite index of $\mathrm{Sl}(2,\mathbb{Z})$, and the induced outer action (up to also passing to an open subgroup of Γ), we essentially find the fundamental groups of all algebraic curves (not necessarily compact) defined over number fields K, and the outer action of $\mathrm{Gal}(\overline{K}/K)$ on them – at least it is true that every such fundamental group appears as a quotient of one of the first groups (*). Taking the "anabelian yoga" (which remains conjectural) into account, which says that an anabelian algebraic curve over a number field K (finite extension of $\mathbb{Q}$) is known up to isomorphism when we know its mixed fundamental group (or what comes to the same thing, the outer action of $\mathrm{Gal}(\overline{K}/K)$ on its profinite geometric fundamental group), we can thus say that all algebraic curves defined over number fields are "contained" in the profinite compactification

Today's focus: genus 1 moduli

$M_{1,1}/\mathbb{Q}$ = fine j-line (elliptic curves' moduli)

$\uparrow$

$M_{1,2}/\mathbb{Q}$ = universal elliptic curves

Grothendieck:
truly richness in $\widehat{SL_2(\mathbb{Z})}$

"La Longue Marche à travers la Théorie de Galois"

$\rightsquigarrow$ Nowadays, many inspirations

B. Enriquez defined $\widehat{GT}_{ell}$

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$\mathrm{Sl}\widehat{(2,\mathbb{Z})}$, and in the knowledge of a certain subgroup $\mathbb{\Gamma}$ of its group of outer automorphisms! Passing to the abelianisations of the preceding fundamental groups, we see in particular that all the abelian ℓ-adic representations dear to Tate and his circle, defined by Jacobians and generalised Jacobians of algebraic curves defined over number fields, are contained in this single action of $\mathbb{\Gamma}$ on the anabelian profinite group $\mathrm{Sl}\widehat{(2,\mathbb{Z})}$! ([4])

Universal Weierstrass family of elliptic curves

$$M_{1,2} \rightsquigarrow \mathcal{W} = \{Y^2 = 4X^3 - g_2X - g_3\}/\mathbb{Q} \hookleftarrow \mathrm{Tate}(q) \setminus \{\overset{\vec{w}}{O}\}$$

$$\downarrow \qquad\qquad \downarrow \quad g_2(q) \sim E_4(q), \quad g_3(q) \sim E_6(q)$$

$$M_{1,1} \rightsquigarrow M_{1,\vec{1}} = \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\} \xleftarrow[\vec{q}]{} \mathrm{Spec}\,\mathbb{Q}((q))$$

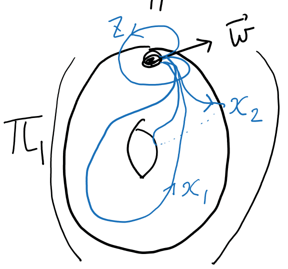
induces

$$1 \longrightarrow \hat{F}_2 \to \pi_1(\mathcal{W}, \vec{w}) \longrightarrow \pi_1(M_{1,\vec{1}}, \vec{q}) \longrightarrow 1$$

$\hat{F}_2$ =

$x_1 x_2 x_1^{-1} x_2^{-1} z = 1$

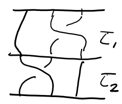
π_1



$\pi_1(M_{1,\vec{1}}, \vec{q})$ = $\hat{B}_3 \rtimes G_{\mathbb{Q}}$

$\downarrow$

$\widehat{SL_2(\mathbb{Z})} \rtimes G_{\mathbb{Q}}$

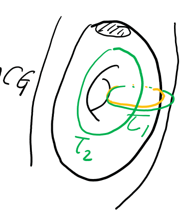


(3 strings braids)

$\tau_1 \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2$

$\tau_1 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\quad \tau_2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

MCG



Universal monodromy representation found in the form (B. Enriquez 2014)

$$\varphi_{1,\vec{1}} : \pi_1(M_{1,\vec{1}}) \hookrightarrow \widehat{GT}_{ell} \longrightarrow \mathrm{Aut}\,\hat{F}_2$$

with split extension $\underline{\widehat{GT}_{ell}} = \hat{B}_3 \rtimes \underline{\widehat{GT}}$

arithmetic functions $\chi(\sigma)$, f_σ

B_σ, $\rho_m(\sigma)$, $\kappa_m^{(a)}(\sigma)$

<u>Proposition</u> (N.)

(1) One has $\rho_\Delta : \widehat{GT}_{ell} \longrightarrow \hat{\mathbb{Z}}$ s.t. $\rho_\Delta(\tau_1) = \rho_\Delta(\tau_2) = -1$ $\quad \Delta = g_2^3 - 27g_3^2$

monodromy action on $\{\sqrt[N]{\Delta}\}$

(2) Meta-abelian reduction

$$\widehat{GT}_{ell} \xrightarrow[\varphi_{1,\vec{1}}]{} \mathrm{Aut}(\hat{F}_2) \longrightarrow \mathrm{Aut}(\hat{F}_2/\hat{F}_2'')$$

can be encoded in combinatorial function in the form

$$\mathbb{E} : \widehat{GT}_{ell} \times (\mathbb{Q} \otimes \hat{\mathbb{Z}}^2) \longrightarrow \hat{\mathbb{Z}} \quad \text{(Adelic Eisenstein invariant)}$$

<u>NB</u>. Let $\rho_{ab} = \tilde{\varphi}_{1,\vec{1}}^{ab} : \widehat{GT}_{ell} \longrightarrow \mathrm{Aut}(\hat{F}_2^{ab}) = GL_2(\hat{\mathbb{Z}})$

$$\sigma \longmapsto \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

On $\mathrm{Ker}(\rho_{ab})$, one has a nice mapping (Bloch, Tsunogai [N1995]) U. Tokyo

$$\mathfrak{E} : \mathrm{Ker}(\rho_{ab}) = C\hat{B}_3 \rtimes G_{\mathbb{Q}(\zeta_\infty)} \longrightarrow \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]]_+$$

but its good extension to $\hat{B}_3 \rtimes G_{\mathbb{Q}} = \pi_1(M_{1,\vec{1}})$ was obtained in [N2013] PRIMS in the form

$$\{\mathbb{E}_m : \pi_1(M_{1,\vec{1}}) \times \hat{\mathbb{Z}}^2 \longrightarrow \hat{\mathbb{Z}}\}_{m \geq 1}$$

Afterwards, we found
- composition law (Repère Mobile) cf [N2019] Ihara 80
- homogeneity ($\mathbb{E}'_{km}(\sigma, ku) = \mathbb{E}'_m(\sigma, u)$)

and
- extension to $\widehat{GT}_{ell}$ (using "$\widehat{GT} = \widehat{GTK}$" [Enriquez 2007], cf. [N2018] MFO-Report)

Put all together gave

$$\mathbb{E} : \widehat{GT}_{ell} \times (\mathbb{Q} \otimes \hat{\mathbb{Z}}^2) \longrightarrow \hat{\mathbb{Z}}.$$

3 Profinite 3-braid analysis by Eisenstein invariants

$$\mathbb{E}(\sigma, u) \quad \left(\sigma \in \hat{B}_3,\ u \in \mathbb{Q}\otimes\hat{\mathbb{Z}}^2\right)$$

Recall we have $\rho_{ab} : \hat{B}_3 \twoheadrightarrow SL_2(\hat{\mathbb{Z}}) \subset \mathrm{Aut}(\hat{F}_2^{ab})$

$$\sigma \longmapsto \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

Write $\mathbb{E}_\sigma(u) = \mathbb{E}(\sigma, u)$ and for a fixed $\sigma \in \hat{B}_3$

Consider the function

$$\mathbb{E}_\sigma : (\mathbb{Q}\otimes\hat{\mathbb{Z}})^2 \longrightarrow \hat{\mathbb{Z}}.$$

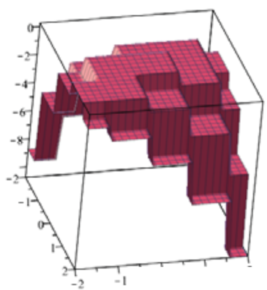
Let us recall $\hat{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$. in $\mathbb{Q}\otimes\hat{\mathbb{Z}}$

Proposition Fix $\sigma \in \hat{B}_3$, and consider the restrictions $\left\{\begin{array}{l} \mathbb{E}_\sigma|_{\hat{\mathbb{Z}}^2} : \hat{\mathbb{Z}}^2 \\ \mathbb{E}_\sigma|_{\mathbb{Q}^2} : \mathbb{Q}^2 \end{array}\right. \rightarrow \hat{\mathbb{Z}}$

① $\mathbb{E}_\sigma : \hat{\mathbb{Z}}^2 \to \hat{\mathbb{Z}}$ is determined by $\rho_{ab}(\sigma) \in SL_2(\hat{\mathbb{Z}})$

In fact, $\exists$ quadratic form $Q_\sigma : \hat{\mathbb{Z}}^2 \to \hat{\mathbb{Z}}$
& linear form $l_\sigma : \hat{\mathbb{Z}}^2 \to \hat{\mathbb{Z}}$

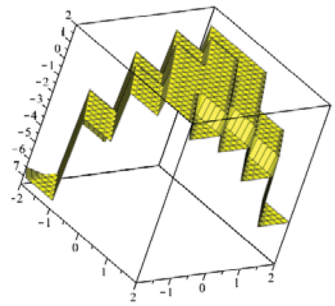
s.t. $\mathbb{E}_\sigma = -\frac{1}{2}(Q_\sigma + l_\sigma)$



② Conversely, $\rho_{ab}(\sigma) \in SL_2(\hat{\mathbb{Z}})$ is determined by $\mathbb{E}_\sigma : \hat{\mathbb{Z}}^2 \to \hat{\mathbb{Z}}$

③ If $\sigma \in B_3$, then $\mathbb{E}_\sigma : (\mathbb{Q}\otimes\hat{\mathbb{Z}})^2 \to \hat{\mathbb{Z}}$ restricts to

$$\mathbb{E}_\sigma : \mathbb{Q}^2 \to \mathbb{Z}$$



Application to Rademacher function

Dedekind eta function $\eta(\tau) = q_\tau^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q_\tau^n)$ $\quad q_\tau = e^{2\pi i \tau}$ $(\tau \in \mathbb{C}, \operatorname{Im}\tau > 0)$

has transformations

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)\sqrt{c\tau+d}\ \eta(\tau) \qquad \varepsilon \in \mu_{24}$$

There are variations of choices to give $\varepsilon = e^{\frac{2\pi i}{24}\Psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$

$$\Psi : SL_2(\mathbb{Z}) \longrightarrow \mathbb{Z} \qquad (\text{relying branches of } \sqrt{c\tau+d} \ \& \ \text{mod } 24)$$

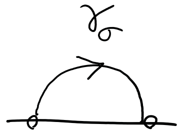
Rademacher - Grosswald (1972 book)

$$\Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \dfrac{b}{d} & c = 0 \\ \dfrac{a+d}{c} - 12\,\operatorname{sgn}(c) \displaystyle\sum_{\mu=1}^{|c|-1} \left(\left(\frac{a\mu}{|c|}\right)\right)\left(\left(\frac{\mu}{|c|}\right)\right) - 3\,\operatorname{sgn}(c(a+d)) & c \neq 0 \end{cases}$$

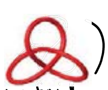
factors through $PSL_2(\mathbb{Z})/_{\text{conj}} \longrightarrow \mathbb{Z}$

E. Ghys (Modular knots) ICM 2006

A hyperbolic matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$ $\quad (|a+d| > 2)$

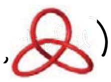


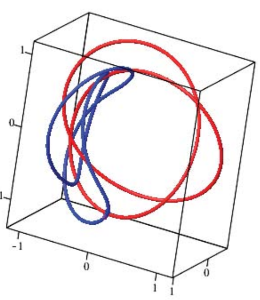
induces an infinite geodesic γ_σ on $\{\operatorname{Im}\tau > 0\} = \mathfrak{h}$

and a closed loop l_σ in $\pi_1(S^3 -$  $) = B_3 = \pi_1(M_{1,\vec{1}}(\mathbb{C})) \ni (g_2, g_3)_{\gamma_\sigma}$

trefoil

that satisfies

$\operatorname{link}(l_\sigma,$  $) = \Psi(\sigma).$



$$\sigma = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$$

$$\Psi(\sigma) = -1$$

Application : Profinite Rademacher function(s)

Thm (1) The mapping $\sigma \longmapsto \rho_\Delta(\sigma) + 12\,\mathbb{E}(\sigma, (\frac{1}{m}, 0))$
factors through $\hat{B}_3 \longrightarrow \widehat{SL_2(\mathbb{Z})}$; defines
$\{ \hat{\Psi}_m : \widehat{SL_2(\mathbb{Z})} \longrightarrow \hat{\mathbb{Z}} \}_{m \geq 1}$

(2) If $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{Z}) \subset \widehat{SL_2(\mathbb{Z})}$
and if $m > a > 0$, $m > |c|$, then $\hat{\Psi}_m(\sigma) = \Psi(\sigma)$.

In fact, for $\sigma \in SL_2(\mathbb{Z})$
$$\hat{\Psi}(\sigma) = \lim_{m \to \infty} \hat{\Psi}_m(\sigma) \quad \text{is well-defined.}$$

(Sketch) Let $\mathbb{E}(\sigma) := \mathbb{E}(\sigma, (\frac{1}{m}, 0))$ for $\sigma \in \hat{B}_3$ and compare two maps
$$\rho_\Delta, \mathbb{E} : \hat{B}_3 \longrightarrow \hat{\mathbb{Z}}$$

(1) We have $1 \longrightarrow \langle (\tau_1\tau_2)^6 \rangle \to \hat{B}_3 \longrightarrow \widehat{SL_2(\mathbb{Z})} \longrightarrow 1$ central ext $(*)$

and $\begin{cases} \rho_\Delta(\sigma (\tau_1\tau_2)^{6k}) = \rho_\Delta(\sigma) - 12k & \text{(homomorphism)} \\ \mathbb{E}(\sigma(\tau_1\tau_2)^{6k}) = \mathbb{E}(\sigma) + k & \text{(composition law)} \end{cases}$

hence $\hat{\Psi}_m = \rho_\Delta + 12\,\mathbb{E}$ factors through $\widehat{SL_2(\mathbb{Z})}$

(2) explicit computation of $\mathbb{E}_m(\sigma, u)$ for $\sigma \in \cancel{SL_2(\mathbb{Z})}$ B_3, $u \in \mathbb{Z}^2$
($\exists$ generalized Dedekind sum formula)

NB. The above central extension has a factoring set in $H^2(\widehat{SL_2(\mathbb{Z})}, \langle (\tau_1\tau_2)^6 \rangle) \cong \mathbb{Z}/12\mathbb{Z}$
defined by the (set-th) section $s : \widehat{SL_2(\mathbb{Z})} \to \hat{B}_3$ s.t. $\mathbb{E}(s(\bar{\sigma})) = 0$.
Regarding $\langle (\tau_1\tau_2)^6 \rangle \cong 12\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$, the above 2-cochain $\mu(\bar{\sigma}, \bar{\tau}) = s(\sigma) + s(\tau) - s(\sigma\tau)$
has a unique 1-cochain $\varphi \in C^1(\widehat{SL_2(\mathbb{Z})}, \hat{\mathbb{Z}})$. Our $\hat{\Psi}_m$ does the role of φ.
s.t. 12μ is a coboundary from $12\varphi \in C^1(\widehat{SL_2(\mathbb{Z})}, 12\hat{\mathbb{Z}})$.

[4] Examples (Lissajous 3-braids)

jt w/ E. Kin, H. Ogawa (ArXiv 2008.00585)

Lissajous plane curve

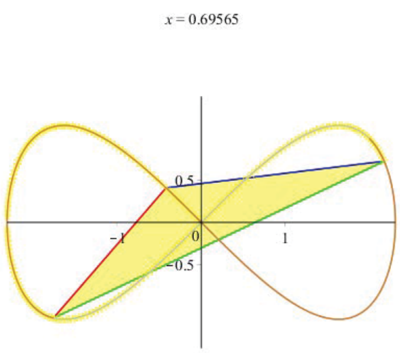
$$L(t) = \sin(2\pi m t + \delta) + i \sin(2\pi n t)$$

3-body motion $t \in [0, \frac{1}{3}]$ (δ : phase difference)

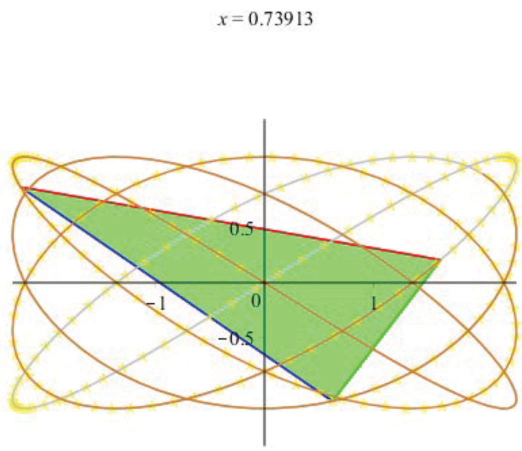
$$\left\{\begin{array}{l} a(t) = L(t - \frac{1}{3}) \\ b(t) = L(t) \\ c(t) = L(t + \frac{1}{3}) \end{array}\right\}$$ defines a braid $\in \hat{B}_3$ iff (m, n) : collision-free

We have a good classification in terms of level $\in \mathbb{N}$ & slope $\in \mathbb{Q}_+$

m=1, n=-2, delta=0



m=4, n=-5, delta=0



> *### RIMS2021_June29_presentation*

> $A := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$A := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad (1)$$

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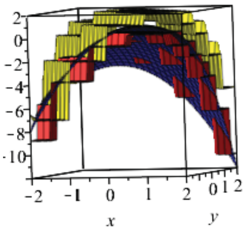
"red=on Z^2,yellow=on Q^2"



$$\text{"rho\_GL(sigma)=",} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

"m=", 3

$$\begin{bmatrix} -2 & -2 & -2 & -1 & -1 & -1 & -1 \\ -2 & -2 & 0 & -1 & -1 & 0 & -1 \\ -2 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -3 \end{bmatrix} \qquad (2)$$

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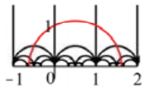
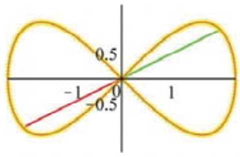
> ##

> *today10cw*(1, 1, 2, 1) [[1, −2, 1], 0], ["abbabaa", "d--p"]

$$\left[X^2 - X\,Y - Y^2, \text{"Eigenvalue=",} \left[\frac{3}{2} + \frac{1}{2}\sqrt{5}, \frac{3}{2} - \frac{1}{2}\sqrt{5}\right], \text{"FixedPt[<=]",} \left[\frac{1}{2} + \frac{1}{2}\sqrt{5}, \frac{1}{2} - \frac{1}{2}\sqrt{5}\right]\right]$$



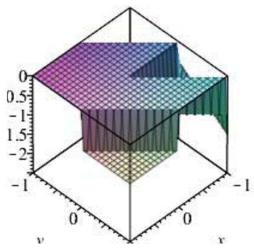
$x = 1.0000$

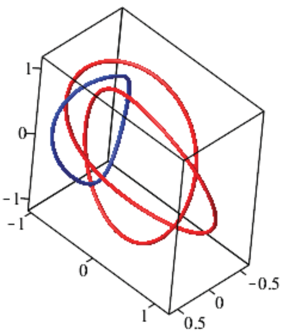


$$\text{"rho\_GL(sigma)=",} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

"m=", 3

$$\begin{bmatrix} -2 & -2 & -2 & -1 & -1 & -1 & -1 \\ -2 & -2 & 0 & -1 & -1 & 0 & -1 \\ -2 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -3 \end{bmatrix}$$





(3)

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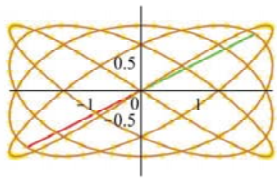
> ##

> *today10cw*(1, 2, 2, 1)

$$[\,[4, -5, 3], 0], [\text{"abbabbabbababbbbabaa", "dbd--pqp"}], \begin{bmatrix} 10 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\left[X^2 - 3\,X\,Y - Y^2, \text{"Eigenvalue=",} \left[\frac{11}{2} + \frac{3}{2}\sqrt{13}, \frac{11}{2} - \frac{3}{2}\sqrt{13}\right], \text{"FixedPt[<=]",} \left[\frac{3}{2} + \frac{1}{2}\sqrt{13}, \frac{3}{2} - \frac{1}{2}\sqrt{13}\right]\right]$$

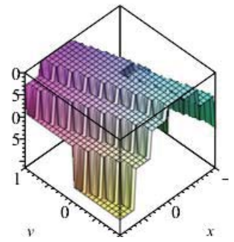
$x = 1.0000$

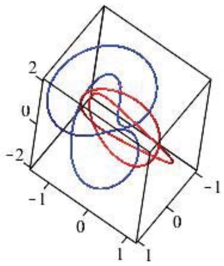


$$\begin{bmatrix} 10 & 3 \\ 3 & 1 \end{bmatrix}$$

"m=", 11

$$\begin{bmatrix} -4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & -3 \end{bmatrix}$$





(4)

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